**Timed Petri Net (TPN)**

TPN = \(<P, T, F, H, W, C, M_0, SI>\)

where:
- \(P\) – a set of places;
- \(T\) – a set of transitions;
- \(F \subseteq (P \times T) \cup (T \times P)\) – a set of arcs from places to transitions and from transitions to places;
- \(W : F \rightarrow \mathbb{N}\) – a weight function of an arc (\(\mathbb{N}\) – the set of natural numbers);
- \(H \subseteq (T \times P)\) – a set of inhibitor arcs (definition of the weight function is omitted, cause \(W : H \rightarrow \{1\}\));
- \(C : P \rightarrow \mathbb{N}\) – a capacity function of a place;
- \(M_0 : P \rightarrow \mathbb{N}\) – an initial marking function;
- \(SI : T \rightarrow \mathbb{Q}_+ \times (\mathbb{Q}_+ \cup \{\infty\})\) – a function assigning a static interval of firing time to every transition (\(\mathbb{Q}_+\) – the set of real non-negative numbers).

Two numbers \(\alpha^S\) and \(\beta^S\), such that \(SI(t) = \langle\alpha^S, \beta^S\rangle\) and \(0 \leq \alpha^S < \infty, \alpha^S \leq \beta^S, \beta^S \leq \infty\), are assigned to every transition \(t\). These numbers define a time interval in which the transition \(t\) must be fired (a time instance, when the transition \(t\) becomes available, is treated as time zero).

**Example 1 – the seasons**

We assume for the Petri Net modelling change of the seasons, that a day is the time unit. Spring lasts 93 days, summer – 93, fall – 90, and winter – 89. The sequence of the seasons is represented by the TPN model in the figure below.

Its formal notation is the following:

\[
P = \{p_0, p_1, p_2, p_3\},
\]

\[
T = \{t_0, t_1, t_2, t_3\},
\]

\[
F = \{(p_0, t_0) \rightarrow 1, \{p_1, t_1\} \rightarrow 1, \{p_2, t_2\} \rightarrow 1, \{p_3, t_3\} \rightarrow 1, \{t_3, p_0\} \rightarrow 1, \{t_0, p_1\} \rightarrow 1, \{t_1, p_2\} \rightarrow 1, \{t_2, p_3\} \rightarrow 1\},
\]

\[
H = \emptyset,
\]

\[
M_0 = \{p_0 = 1, p_1 = 0, p_2 = 0, p_3 = 0\},
\]

\[
SI = \{t_0 \rightarrow <93,93>, t_1 \rightarrow <93,93>, t_2 \rightarrow <90,90>, t_3 \rightarrow <89,89>\}.
\]

In a short form:

\[
SI = \{<93,93>, <93,93>, <90,90>, <89,89>\},
\]

where a position in the set is the transition's number.

**State S** of the Timed Petri Net is denoted by a marking \(M\) and a vector \(I\) (a vector of pairs of non-negative values calculated for every transition, that is enabled to fire in this marking).

\[
S_i = (M_i, I_i), \text{ where for every state the pair } (M_i, I_i) \text{ must be different.}
\]
The transition \( t_i \) is enabled only if:
- every input place contains sufficient number of tokens,
- no input place with an inhibitor arc contains any token,
- the transition \( t_i \) is enabled in the present time (other enabled transitions may be fired later).

Therefore, the conditions (4) – (6) for the transition \( t_i \) must be fulfilled:

(4) \((\forall p)(M(p) \geq F(p,t_i))\)
(5) \((\forall p)((l(t_i,p) = 1) \Rightarrow (M(p) = 0))\)
(6) \(\alpha_i \leq \theta \leq \min\{\beta_k\}\)

where: \( \theta \) – relative time (to the present moment) of firing the transition \( t_i \),
\( t_k \) – other transition enabled to be fired,
\( k \) – range of the set of transitions enabled by \( M \).

Firing the transition \( t_i \) consists of three steps:
- remove \( W(p,t_i) \) tokens from every input place,
- add \( W(t_i,p) \) tokens to every output place,
- calculate a new vector \( I \).

These operations are given by the formula (7):

(7) \((\forall p)(M'(p) = M(p) - W(p,t_i) + W(t_i,p))\)

STEP 1: Remove every pair from the vector \( I \), that is related to transitions, that are not enabled, when \( t_i \) is being fired.

STEP 2: Shift time of every remained firing interval by the value of \( \theta \):

(8) \(\alpha_k = \max\{0, \alpha_k - \theta\}, \quad \beta_k = \beta_k - \theta\)

STEP 3: Introduce pairs of new enabled transitions:

(9) \(\alpha_p = \alpha_p^S, \quad \beta_p = \beta_p^S\)

Example 2

Let us analyse an example from the figure on the right.

The transition \( t_1 \) is enabled to fire, but the firing time \( \theta \) may be e.g. 0; 1,2; 1,46; 1,2345. Hence, we have infinitely many states.

Therefore, we describe the firing time, using inequalities, e.g. \( 0 \leq \theta \leq 3 \).

Instead of the vector \( I \) we have a system of inequalities called a domain, and instead of one state we have a set of states (cause the number \( \theta \) is infinite) called a class.

Classes of states \( C \) are pairs:

\( C_i = (M_i, D_i) \)

where:
- \( M \) – a marking (all states in the class have the same marking),
- \( D \) – a firing domain.
The figure on the right represents a class diagram for the TPN from page 2.

Exercises

Exercise 1.
Describe main differences between PN and TPN.

Exercise 2.
Extend the TPN model from page 1, so that it took into account the leap year.

Exercise 3.
Make a TPN for the PN modelling the control of the traffic light, presented in the figure below (its description is in the instruction to the first laboratory on Petri Nets).

Extend this model according to the following conditions:
• a person presses a button, to get the green light,
• time duration of the green light for a person equals exactly 30 seconds,
• time duration of the green light for cars cannot be shorter than 45 seconds.
Other time parameters should reasonably correspond to reality.

Exercise 4.
Make a TPN for the PN modelling the communication Bond–“M” from ex. 5 from the instruction to the second and third laboratories on Petri Nets.
Propose your own reasonable time parameters, corresponding to reality.