13.
Gauss-Markov whitening, Generalized Least Squares

Noise covariance matrix

$$\mathrm{cov}(Z_N) = \mathbf{E} Z_N Z_N^T$$

is positive definite. We can apply factorization theorem

Twierdzenie 1 *Each symmetric and positive definite matrix* $\mathbf{M}$ *can be written in the form*

$$\mathbf{M} = \mathbf{P}\mathbf{P}^T$$

where $\mathbf{P}$ *is nonsingular (called "root of* $\mathbf{M}$*").*

One can write that

$$\mathrm{cov}(Z_N) = P \cdot P^T = C$$

apply P^{-1} for measurement equation

$$P^{-1} Y_N = P^{-1} \Phi_N \theta + P^{-1} Z_N$$

Introducing

$$\overline{Y}_N = P^{-1} Y_N, \qquad \overline{\Phi}_N = P^{-1} \Phi_N, \qquad \overline{Z}_N = P^{-1} Z_N,$$

we get

$$\overline{Y}_N = \overline{\Phi}_N \theta + \overline{Z}_N$$

What is whitening?

$$\begin{aligned} \mathbf{E}\overline{Z}_N \overline{Z}_N^T &= \mathbf{E} P^{-1} Z_N Z_N^T P^{-1^T} = P^{-1}\left(\mathbf{E} Z_N Z_N^T\right) P^{-1^T} = P^{-1} P \cdot P^T P^{-1^T} = I \\ \mathbf{E}\overline{Z}_N \overline{Z}_N^T &= I \qquad (I \text{ – identity matrix, diagonal – } \overline{Z} \text{ is white}), \end{aligned}$$

GLS method

$$\begin{aligned} \widehat{\theta_{GLS}} &= (\overline{\Phi}_N^T \overline{\Phi}_N)^{-1} \overline{\Phi}_N^T \overline{Y}_N \\ \widehat{\theta_{GLS}} &= (\Phi_N^T P^{-1^T} P^{-1} \Phi_N)^{-1} \Phi_N^T P^{-1^T} P^{-1} Y_N = (\Phi_N^T C^{-1} \Phi_N)^{-1} \Phi_N^T C^{-1} Y_N \end{aligned}$$

Advantage: It is B.L.U.E.

Disadvantage: Alternate estimation of system and the colour C is necessary.