11.
Best Linear Unbiased Estimate

1. LS properties for $N < \infty$

bias

$$\begin{aligned} \text{let } L_N &= \left(X_N^T X_N\right)^{-1} X_N^T \text{ we get } \mathbf{a}_N = L_N Y_N \text{ (\textbf{linear estimate})} \\ \mathbf{a}_N &= L_N Y_N = L_N(X_N \mathbf{a}^* + Z_N) = \mathbf{a}^* + L_N Z_N \\ \mathbf{E}\mathbf{a}_N &= \mathbf{a}^* + \mathbf{E}\left\{L_N\right\} \mathbf{E}\left\{Z_N\right\} = \mathbf{a}^* \end{aligned}$$

conclusion: $\mathbf{a}_N$ is unbiased for both random and deterministic input

covariance matrix

$$\begin{aligned} cov(\mathbf{a}_N) &= \mathbf{E}\left\{(\mathbf{a}_N - \mathbf{a}^*)(\mathbf{a}_N - \mathbf{a}^*)^T\right\} \\ \mathbf{a}_N - \mathbf{a}^* &= L_N Z_N \\ cov(\mathbf{a}_N) &= \mathbf{E}\left\{L_N Z_N Z_N^T L_N^T\right\} = L_N \left(\mathbf{E} Z_N Z_N^T\right) L_N^T \end{aligned}$$

structure

$$\mathbf{E} Z_N Z_N^T = \begin{bmatrix} \mathbf{E}z_1^2 & \mathbf{E}z_1 z_2 & .. & \mathbf{E}z_1 z_N \\ \mathbf{E}z_2 z_1 & \mathbf{E}z_2^2 & .. & : \\ : & : & : & : \\ \mathbf{E}z_N z_1 & .. & .. & \mathbf{E}z_N^2 \end{bmatrix} = \sigma_z^2 I$$

thus

$$cov(\mathbf{a}_N) = \sigma_z^2 L_N L_N^T$$

for $L_N L_N^T$ we get

$$L_N L_N^T = \left(X_N^T X_N\right)^{-1} X_N^T X_N \left(X_N^T X_N\right)^{-1} = \left(X_N^T X_N\right)^{-1}$$

hence

$$cov(\mathbf{a}_N) = \sigma_z^2 \left(X_N^T X_N\right)^{-1}$$

orthogonal planing
when $X_N^T X_N$ is diagonal (active experiment)

1.1. Optimality of $\mathbf{a}_N$ in LUE class

LUE class

$$\{\alpha_N : \alpha_N = M_N Y_N, \quad \mathbf{E}\alpha_N = \mathbf{a}^*, \quad M_N - \text{deterministic}\}$$

expected value

$$\mathbf{E}\alpha_N = \mathbf{E}M_N Y_N = \mathbf{E}M_N X_N \mathbf{a}^* + \mathbf{E}M_N Z_N \text{ conclusion: } M_N X_N = I$$

covariance matrix

$$\begin{aligned} cov(\alpha_N) &= \mathbf{E}\left\{M_N Z_N Z_N^T M_N^T\right\} = M_N \left(\mathbf{E}Z_N Z_N^T\right) M_N^T \\ cov(\alpha_N) &= \sigma_z^2 M_N M_N^T \end{aligned}$$

we will prove that

$$cov(\mathbf{a}_N) \leqslant cov(\alpha_N)$$

what means "$\leqslant$"

Definicja 1 *Matrix $A \in R^{n,n}$ is nonnegative definite (i.e. $A \geqslant 0$), if for each vector $w = R^n$ it holds that*

$$w^T A w \geqslant 0$$

Twierdzenie 1 *Matrix of the for AA^T is nonnegative for any A.*

let us pu $A = M_N - L_N$

$$(M_N - L_N)(M_N - L_N)^T = (M_N - L_N)(M_N^T - L_N^T) = M_N M_N^T - M_N L_N^T - L_N M_N^T + L_N L_N^T = (***)$$

and notice that

$$\begin{aligned} M_N L_N^T &= M_N X_N \left(X_N^T X_N\right)^{-1} = I \left(X_N^T X_N\right)^{-1} = \left(X_N^T X_N\right)^{-1} = L_N L_N^T \\ L_N M_N^T &= (M_N L_N^T)^T = L_N L_N^T \end{aligned}$$

hence

$$(***) = M_N M_N^T - L_N L_N^T$$

and by above theorem we get

$$\begin{aligned} M_N M_N^T - L_N L_N^T &\geqslant 0 \qquad / \cdot \sigma_z^2 \\ cov(\alpha_N) - cov(\mathbf{a}_N) &\geqslant 0 \end{aligned}$$