3.
Convergence of random sequences

1. Types of convergence

Deterministic sequence

$$\lim_{k\to\infty} a_k = g \Leftrightarrow \bigwedge_{\varepsilon>0} \bigvee_{k_0} \bigwedge_{k>k_0} |a_k - g| < \varepsilon$$

Rate of convergence

notation $o()$ – "faster than"

$$a_k = o(b_k) \Leftrightarrow \lim_{k\to\infty} \frac{a_k}{b_k} = 0 \text{ (and both sequences tends to zero)}$$

notation $O()$ – "the same rate"

$$a_k = O(b_k) \Leftrightarrow \bigvee_{c<\infty} |a_k| \leqslant c\,|b_k|$$

Sequences of random variables $\{\varkappa_k\}$

notation $\lim_{k\to\infty}$ cannot be applied here, because $|a_k - g| < \varepsilon$ is the random event

Definicja 1 *The sequence of random variables $\{\varkappa_k\}$ converges **in probability** (weakly) to $\varkappa^{\#}$ as $k \to \infty$ if for each $\varepsilon > 0$ it holds that*

$$\lim_{k\to\infty} P(\left|\varkappa_k - \varkappa^{\#}\right| > \varepsilon) = 0, \text{ or equivalently } \lim_{k\to\infty} P(\left|\varkappa_k - \varkappa^{\#}\right| < \varepsilon) = 1$$

The value $\varkappa^{\#}$ is called stochastic limit of $\{\varkappa_k\}$

$$P \lim_{k\to\infty} \varkappa_k = \varkappa^{\#} \tag{1}$$

Definicja 2 *The sequence of random variables $\{\varkappa_k\}$ converges **with probability 1** (strongly) to $\varkappa^*$ as $k \to \infty$ if*

$$P(\lim_{k\to\infty} \varkappa_k = \varkappa^*) = 1$$

Lemat 1 *Convergence with probability 1 (strong) implies convergence in probability (weak).*

Definicja 3 *The sequence of random variables* $\{\varkappa_k\}$ *converges* ***in the mean of order*** r *to the limit* $\varkappa^*$ *as* $k \to \infty$ *if*

$$\lim_{k\to\infty} \mathbf{E}\,|\varkappa_k - \varkappa^*|^r = 0,$$

in particular, in the ***mean sqeare sense****, if*

$$\lim_{k\to\infty} \mathbf{E}(\varkappa_k - \varkappa^*)^2 = 0$$

Definicja 4 *The sequence of randomvariables* $\{\varkappa_k\}$ *has* ***has the rate of order*** $O(e_k)$ ***in probability*** *as* $k \to \infty$ *(i.e. asymptotically), where* $\{e_k\}$ *is a deterministic number seqence convergent to zero, i.e.*

$$\varkappa_k = O(e_k) \text{ in probability}$$

if and only if $\left\{\frac{\varkappa_k}{e_k}\chi_k\right\}$ *tends to zero in probability for each sequence* $\{\chi_k\}$*, such that* $\lim_{k\to\infty} \chi_k = 0$*.*

Definicja 5 *Ciąg zmiennych losowych* $\{\varkappa_k\}$ *ma* ***szybkość zbieżności rzędu*** $O(e_k)$ ***według średniej z kwadratem*** *przy* $k \to \infty$ *jeżeli istnieje stała* $0 \leq c < \infty$*, taka, że*

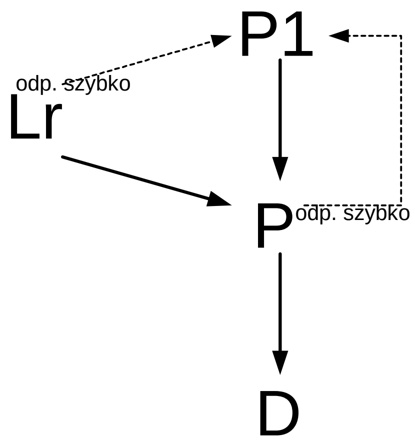
$$\mathbf{E}\varkappa_k^2 \leq c e_k$$

Lemat 2 *Jeżeli* $\varkappa_k = O(e_k)$ *według średniej z kwadratem, to* $\varkappa_k = O(\sqrt{e_k})$ *według prawdopodobieństwa.*

Definicja 6 *The sequence of random variables* X_k *converges in distribution to the random variable* X*, if*

$$\lim_{k\to\infty} F_k(x) = F(x)$$

2. Relations between various types of convergence



Proof of the fact that $P1 \Longrightarrow P$

obviously from $P(\lim_{k\to\infty} \varkappa_k = \varkappa^*) = 1$, we conclude that $\sum_{k=1}^{\infty} P(|\varkappa_k - \varkappa^*| < \varepsilon) < \infty$ and $P(|\varkappa_k - \varkappa^*| < \varepsilon) \to 0$ as $k \to \infty$

Proof of the fact that $Lr \Longrightarrow P$

from definition

$$\mathbf{E}\,|\varkappa_k - \varkappa^*|^r = \int_{\Omega} |\varkappa_k - \varkappa^*|^r\, d\omega \geqslant \int_{\{|\varkappa_k - \varkappa^*| > \varepsilon\}} |\varkappa_k - \varkappa^*|^r\, d\omega \geqslant \varepsilon^r \int_{\{|\varkappa_k - \varkappa^*| > \varepsilon\}} d\omega = \varepsilon^r P(|\varkappa_k - \varkappa^*| > \varepsilon)$$

thus

$$P(|\varkappa_k - \varkappa^*| > \varepsilon) \leqslant \frac{1}{\varepsilon^r}\mathbf{E}\,|\varkappa_k - \varkappa^*|^r$$

in particular, for $r = 2$ and $\varkappa^* = \mathbf{E}\varkappa$

$$P(|\varkappa - \mathbf{E}\varkappa| > \varepsilon) \leqslant \frac{1}{\varepsilon^2} var\varkappa$$

Proof of the fact that: $\sum_{k=1}^{\infty} P(|\varkappa_k - \varkappa^*| < \varepsilon) < \infty$ and $\varkappa_k \xrightarrow{p} \varkappa^* \Longrightarrow \varkappa_k \xrightarrow{p1} \varkappa^*$

$$P(\sup_{k\geqslant k_0}|\varkappa_k - \varkappa^*| \ > \ \varepsilon) = P(|\varkappa_k - \varkappa^*| > \varepsilon \text{ for some particular } k \geqslant k_0) = P\left(\bigcup_{k=k_0}^{\infty}(|\varkappa_k - \varkappa^*| > \varepsilon)\right) \leqslant$$

$$\leqslant \sum_{k=k_0}^{\infty} P(|\varkappa_k - \varkappa^*| > \varepsilon) \to 0, \text{ (since the sum of infinite elements is finite) as } k_0 \to \infty$$

Example

if $P(|\varkappa_k - \varkappa^*| < \varepsilon) = O(\frac{1}{k})$ then $\varkappa_k \xrightarrow{p} \varkappa^*$, but not $\varkappa_k \xrightarrow{p1} \varkappa^*$

Problem

let $\varkappa_k \xrightarrow{p} \varkappa^*$ as $k \to \infty$, can we conclude that $g(\varkappa_k) \xrightarrow{p} g(\varkappa^*)$, as $k \to \infty$???

Yes – provided that $g()$ is continuous in the point $\varkappa^*$

3. Strong Law of Large Numbers (Kolmogorov)

Assumptions

(a) $X_1, X_2, ..., X_N$ – the sequence of independent and identically distributed random variables

(b) exist $\mathbf{E}X_i = m < \infty$

Thesis

$$\frac{1}{N}\sum_{i=1}^{N} X_i \xrightarrow{p1} m, \text{ as } N \to \infty$$

other versions of SLLN – see [Feller], [Krzyśko], [Ninness]

4. Strong Law of Large Nubmers (with relaxed assumptions)

Assumptions

(a) $X_1, X_2, ..., X_N$ – independent, but in general of different distributions

(b) exist $\mathbf{E}X_i = m_i < \infty$

(c) exist $varX_i = \sigma_i^2 < \infty$

(d) $\sum_{i=1}^{\infty} \frac{\sigma_i^2}{i^2} < \infty$

Thesis

$$\frac{1}{N}\sum_{i=1}^{N} X_i - \frac{1}{N}\sum_{i=1}^{N} m_i \xrightarrow{p1} 0, \text{ as } N \to \infty$$

5. Central Limit Theorem (Lindenberg–Levy)

Assumptions

(a) $X_1, X_2, ..., X_N$ – i.i.d.

(b) $\mathbf{E}X_i = m < \infty$

(c) $var X_i = \sigma^2 < \infty$

Thesis

$$\frac{\sum_{i=1}^{N} X_i - Nm}{\sigma\sqrt{N}} \xrightarrow{D} \mathcal{N}(0,1), \text{ as } N \to \infty$$

Conclusion

$$\frac{\frac{1}{N}\sum_{i=1}^{N} X_i - m}{\frac{\sigma}{\sqrt{N}}} \xrightarrow{D} \mathcal{N}(0,1) \qquad \frac{1}{N}\sum_{i=1}^{N} X_i \xrightarrow{D} \mathcal{N}(m, \frac{\sigma^2}{N})$$

Barry-Essena inequality

let $\varkappa_N = \frac{\sum_{i=1}^{N} X_i - Nm}{\sigma\sqrt{N}}$

$$\sup_x |F_{\varkappa_N}(x) - \Phi(x)| \leqslant \frac{33}{4}\frac{\mathbf{E}\left|X_i - m\right|^3}{\sigma^3\sqrt{N}} = O\left(\frac{1}{\sqrt{N}}\right)$$