2.
Rejection method

Lemma 1. If $X\tilde{}f(x)$ and $U\tilde{}\mathcal{U}[0,1]$ then the pair (X,Y), were $Y=cf(X)U$, and c is any constant, i.e.

$$(X, cf(X)U)$$

is uniformly distributed on the set

$$A=\{(x,y): x\in R, y\in[0,cf(x)]\}$$

Proof

conditional distribution (fixed X)

$$F_{y|x}(\alpha)=P(Y\leqslant\alpha|X=x)=P(cf(X)U\leqslant\alpha|X=x)=P(cf(x)U\leqslant\alpha)=P(U\leqslant\frac{\alpha}{cf(x)})=\frac{\alpha}{cf(x)}$$

conditional probability density dunction

$$f_{y|x}(\alpha)=\frac{\partial F_{y|x}(\alpha)}{\partial\alpha}=\frac{1}{cf(x)}\text{ -- conditional distribution is uniform}$$

let us take any set $B\subseteq A$

$$P((X,Y)\in B)=P(B)=\iint_B f(x,y)dxdy=[\text{Bayes theorem}]=\iint_B f_{y|x}(\alpha)f(x)dxd\alpha=\frac{1}{c}\iint_B dxdy=\frac{1}{c}\mu(B)$$

in particular, for $B=A$

$$P((X,Y)\in A)=1=\frac{1}{c}\mu(A),\text{ hence } c=\mu(A)$$

conclusion

$$P(B)=\frac{\mu(B)}{\mu(A)}\text{ -- uniform on the whole set } A$$

Lemma 2. If the pair of random variables (X,Y) is uniformly distributed on the set

$$A=\{(x,y): x\in R, y\in[0,cf(x)]\}$$

then X has the probability density function equal to $f(x)$.

Proof

let D be any subset of R

$$D \subseteq R$$

and B_D – the set of pairs (x, y), such that $x \in D$

$$B_D = \{(x, y) : x \in D, y \in [0, cf(x)]\} \subseteq A$$

distribution of X has the property that

$$P(X \in D) = P(D) = P((X, Y) \in B_D) = P(B_D) = [\text{uniform}] = \frac{\mu(B_D)}{\mu(A)} = \frac{\int_D cf(x)dx}{\int_R cf(x)dx} = \int_D f(x)dx.$$

Scheme of the method

$f(x)$ – p.d.f. of X, (we want to generate)

$g(x)$ – supporting p.d.f. (easy to generate), it exists $c > 0$, such that

$$f(x) \leqslant cg(x) \text{ for each } x \in R$$

Step 1. Generation

– generate realizations x of random variable X having the p.d.f. $g(x)$ (e.g. by the inverse method)

– generate realizations u of random variables U from tha uniform distribution $\mathcal{U}[0, 1]$

– create the pairs $(x, y) = (x, cg(x)u)$, which have uniform distribution on the set $A_g = \{(x, y) : x \in R, y \in [0, cg(x)]\}$

Step 2. Rejection

– reject the pairs (x, y) from *Step 1*, which does not belong to $A_f = \{(x, y) : x \in R, y \in [0, f(x)]\}$, i.e. leave (select) the pairs which fulfills the following condition

$$cg(x)u \leqslant f(x)$$

Conclusion – we obtain accurate generator (no approximation).

Applicability conditions

1) for the function $f(x)$ we can propose proper $g(x)$

2) $g(x)$ is easy to generate

3) probability of rejection should be relatively small

$$P(\text{selection } (x,y)) = \frac{\mu(A_f)}{\mu(A_g)} = \frac{\int_{-\infty}^{\infty} f(x)dx}{c\int_{-\infty}^{\infty} g(x)dx} = \frac{1}{c}$$

$$\text{we postulate } \frac{1}{c} \rightarrow \max\,,\ \text{but } c \geqslant 1$$

Accurate generation of normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad\qquad g(x) = \frac{1}{2}e^{-|x|} \text{ – Laplace distribution}$$

one can show that

$$f(x) \leqslant cg(x), \text{ where } c = \sqrt{\frac{2e}{\pi}}$$

selection condition

$$\sqrt{\frac{2e}{\pi}}\frac{1}{2}e^{-|x|}u \leqslant \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

$$\cdots$$

$$(|x|-1)^2 \leqslant -2\ln u$$

Comment

before rejection/selection x's have Laplace distribution, and after rejection/selection $x\tilde{}\mathcal{N}(0,1)$