1.
Generation of random numbers (inverse method)

1. How to obtain pseudo-random numbers

nonlinear feedback

$$x_k = \mathcal{F}(x_{k-1}, x_{k-2}, ..., x_{k-q})$$

Postulates concerning $\mathcal{F}$:

1) long period of the sequence $\{x_k\}$

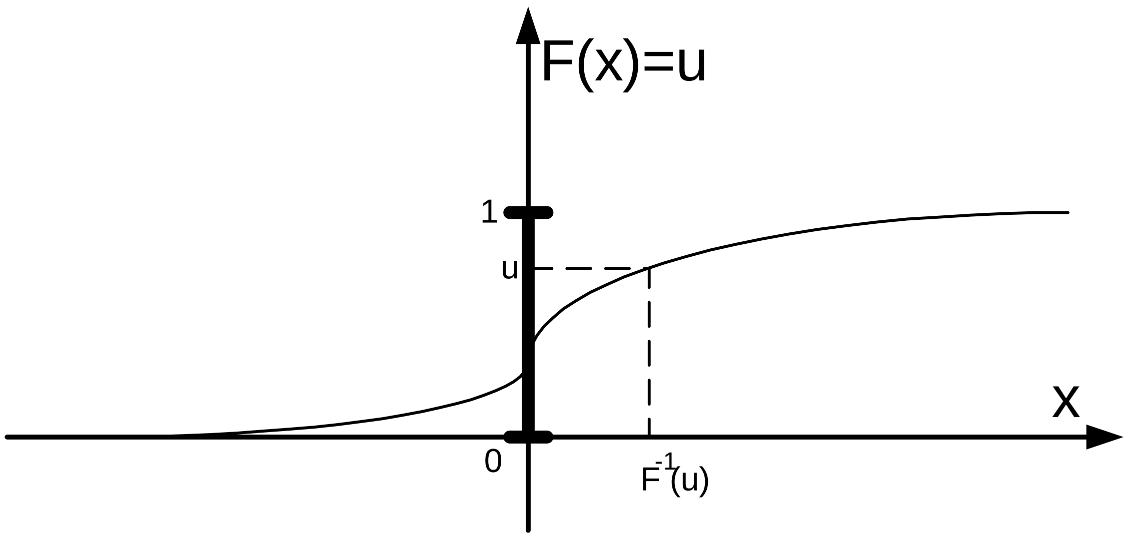
2) regular distribution of x_k

2. Inverse method

Problem formulation

– we have high quality generator of uniformly distributed random variable $u\tilde{}\ \mathcal{U}[0,1]$

– we want to generate realizations of random variable X of given probability density function $f(x)$, such that the distribution function $F(x) = P(X \leqslant x) = \int_{-\infty}^{x} f(x)dx$ is strictly monotonous and its inversion has analytic form



$F^{-1}()$ exists and is well defined

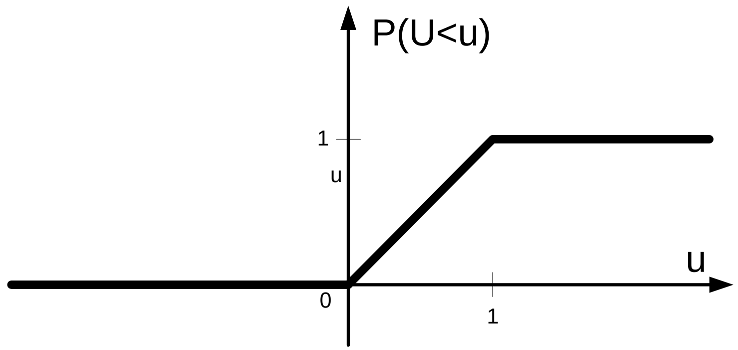
$$
\begin{aligned}
u &= F(x), \qquad u \in [0,1] \\
x &= F^{-1}(u), \qquad x \in R
\end{aligned}
$$

Lemma 1

If $U\tilde{\ }\mathcal{U}[0,1]$, then $X = F^{-1}(U)$ has distribution function $F(X)$ (i.e. its probability density function is $f(x)$).

Proof

$P\left\{X \leqslant x\right\} = P\left\{F^{-1}(U) \leqslant x\right\} = P\{U \leqslant F(x)\} = (...)$



$(...) = F(x)$

Lemma 2

If $X\tilde{\ }F(x)$, then $U = F(X)\tilde{\ }\mathcal{U}[0,1]$.

Proof

$P(U \leqslant u) = P(F(X) \leqslant u) = P(X \leqslant F^{-1}(u)) = F(F^{-1}(u)) = u$

Scheme of the method

1) generate $u\tilde{\ }\mathcal{U}[0,1]$

2) for the designed p.d.f. $f(x)$ compute $F()$ and its inverse $F^{-1}()$

3) compute $F^{-1}(u)$

Example

exponential distribution

$$
\begin{aligned}
f(x) &= \begin{cases} \alpha e^{-\alpha x}, \text{ as } x \geqslant 0 \\ 0, \text{ as } x < 0 \end{cases} \\
F(x) &= \int_0^x \alpha e^{-\alpha x} dx = [-e^{-\alpha x}]_0^x = 1 - e^{-\alpha x} = u \\
e^{-\alpha x} &= 1 - u \\
-\alpha x &= \ln(1-u) \\
x &= -\frac{\ln(1-u)}{\alpha}
\end{aligned}
$$

please notice that

$$-\frac{\ln u}{\alpha} \text{ hasthe same distribution}$$

Exercices

triangular distribution

Laplace distribution

Cauchy/Lorenz distribution