

# Identification of Interconnected Systems by Instrumental Variables Method

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**Abstract**—The paper addresses the problem of parameter estimation of elements in complex, interconnected systems. Similarity between causes of biases in the least squares estimates for a simple *SISO* linear dynamic object, and for a *MIMO* linear static system with composite structure, was noticed in the paper. For linear complex static system, the instrumental variable estimate was proposed and compared with the least squares approach. The strong consistency of the presented parameter estimate was proved. Also the optimal values of instrumental variables were established, and the method of their suboptimal generation was presented. The conclusions were verified in numerical experiments.

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## I. INTRODUCTION

We consider the problem of parameter estimation in complex, interconnected systems with the presence of random noises. In a lot of commonly met hierarchical control problems, the accurate mathematical models of the particular system components are needed. Under the term 'complex' we understand the fact that the system is built of a number of interconnected components (subsystems), e.g., in the typical production system each element is excited by the outputs of other blocks (see [1]). In consequence of mutual interconnections, the components are dependent and their separation may be impossible or too expensive. In general, excitations of particular element cannot be freely generated in the experiment. It leads to the problem of structural identifiability (i.e. identifiability of separate elements does not imply identifiability of the whole interconnected system [3]) and usually badly conditioned numerical tasks. Moreover, some interaction

signals are hidden, and cannot be directly measured. For these reasons, the algorithms dedicated for single element cannot be directly applied in complex system analysis.

Identifiability of the element, which operates in complex system, depends additionally on the system structure and the values of parameters of other elements. Particularly, the components preceding identified object must guarantee persistency of the input excitation. In the paper we apply and compare two methods – least squares (*l.s.*) and instrumental variables (*i.v.*).

It is commonly known from the linear system theory, that the least squares approach applied for the simple *SISO* linear dynamic object leads to biased estimate. The reason of the bias results from the property of autoregression, i.e. the correlation between the noise and the values of previous outputs of the identified object (see the Appendix). Analogously, for the complex, interconnected systems with random noises, the least squares estimate has the non-zero systematic error even if the number of measurement data tends to infinity. The reason of the bias is that the output noises are transferred to the inputs through the structural feedback.

In the paper, the formal similarity of these problems is shown and the instrumental variables technique, used so far for the linear dynamics identification, was successfully generalized for the systems with complex structure. It is shown that the proposed *i.v.* estimate is strongly consistent independently of the system structure and the color of the noise. Moreover, the computational complexity of the method is comparable with the *l.s.* algorithm. In Section II the identification problem and the purpose is formulated in detail. Next, in Section 3, the properties of the least squares based algorithm proposed in [3] are reminded. In particular, the reason of its bias is shown in detail, and finally, in Section IV the new *i.v.* estimate is introduced and analyzed. Finally, in Section V, the performance of the method is demonstrated by the simulation example.

## II. STATEMENT OF THE PROBLEM

Consider the system shown in Fig. 1. It consists of  $n$  linear elements described as follows

$$y_i = a_i x_i + b_i u_i + \xi_i \quad (i = 1, 2, \dots, n),$$

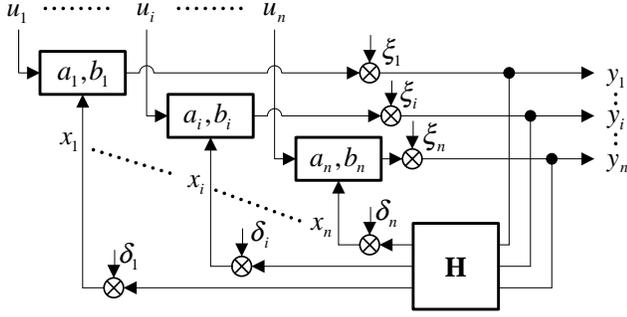


Fig. 1. The complex  $n$ -element linear static system

where

$$\begin{aligned} u &= (u_1, u_2, \dots, u_n)^T \\ x &= (x_1, x_2, \dots, x_n)^T \\ y &= (y_1, y_2, \dots, y_n)^T \end{aligned}$$

are the external inputs, interaction inputs, and system outputs, respectively. The processes

$$\begin{aligned} \delta &= (\delta_1, \delta_2, \dots, \delta_n)^T \\ \xi &= (\xi_1, \xi_2, \dots, \xi_n)^T \end{aligned}$$

are random disturbances. The block  $H$  determines the system structure in the following way

$$x_i = H_i y + \delta_i, \quad (1)$$

where  $H_i$  is the  $i$ th row of the binary matrix  $H$  (i.e.  $H_{i,j} = 0$  – 'no connection',  $H_{i,j} = 1$  – 'is connection').

The aim is to estimate parameters  $\{(a_i, b_i)\}_{i=1}^n$  of the particular elements using the set of data  $\{(u^{(k)}, y^{(k)})\}_{k=1}^N$  collected in the experiment. We emphasize that the internal excitations  $x^{(k)}$  cannot be measured.

We assume that:

(A1) The structure of the system (i.e. the matrix  $H$ ) is known.

(A2) The system is well defined, i.e. for any  $(u, \delta, \xi)$  it exists unique  $y$  (see [3]).

(A3) The noises  $\delta$ , and  $\xi$  are zero-mean, mutually independent, and independent of  $u$ .

(A4) In the noise-free case ( $\delta = 0$ , and  $\xi = 0$ ) the system would be identifiable (see [3]).

(A5) The excitations are rich enough, i.e. the matrix

$$E_N = (e_1, e_2, \dots, e_N),$$

where

$$e = (u^T, \theta^T)^T \text{ and } \theta = A\delta + \xi,$$

is of full rank with probability 1.

Introducing the matrices

$$\begin{aligned} A &= \text{diag}(a_1, a_2, \dots, a_n) \\ B &= \text{diag}(b_1, b_2, \dots, b_n) \\ H &= (H_1^T, H_2^T, \dots, H_n^T)^T \end{aligned}$$

the whole system can be described in the following compact form

$$\begin{cases} y = Ax + Bu + \xi \\ x = Hy + \delta \end{cases}. \quad (2)$$

Inserting  $x$  to the first equation in (2) we obtain

$$\begin{aligned} y &= A(Hy + \delta) + Bu + \xi, \\ (I - AH)y &= Bu + A\delta + \xi, \end{aligned}$$

which leads to

$$y = Ku + G\theta \quad (3)$$

where

$$\begin{aligned} G &= (I - AH)^{-1}, \\ K &= (I - AH)^{-1}B = GB. \end{aligned} \quad (4)$$

The equation (3) resembles description of the object with the input  $u$ , the output  $y$ , the transfer matrix  $K$ , and the noise  $G\theta$ . Invertibility of  $(I - AH)$  in (4) is equivalent to assumption (A2).

### III. LEAST SQUARES APPROACH

Introducing the vectors of input-output data of  $i$ th element

$$\begin{aligned} Y_{iN} &= [y_i^{(1)}, y_i^{(2)}, \dots, y_i^{(N)}], \\ W_{iN} &= [w_i^{(1)}, w_i^{(2)}, \dots, w_i^{(N)}], \end{aligned} \quad (5)$$

$$\text{where } w_i = (x_i, u_i)^T,$$

we obtain the measurement equation

$$Y_{iN} = (a_i, b_i)W_{iN} + \xi_i. \quad (6)$$

Since the input  $x_i$  included in  $w_i$  is unknown (cannot be measured), the least squares estimate cannot be derived directly from (6). Owing to (1) the natural substitution is

$$\tilde{w}_i = (\tilde{x}_i, u_i)^T, \text{ where } \tilde{x}_i = H_i y = x_i - \delta_i.$$

It leads to the following least squares estimate

$$(\hat{a}_i^{l.s.}, \hat{b}_i^{l.s.}) = Y_{iN} \tilde{W}_{iN}^T \left( \tilde{W}_{iN} \tilde{W}_{iN}^T \right)^{-1}, \quad (7)$$

where

$$\tilde{W}_{iN} = [\tilde{w}_i^{(1)}, \tilde{w}_i^{(2)}, \dots, \tilde{w}_i^{(N)}].$$

*Remark 1:* The estimate (7) originates from the modified version of measurement equation (6), in which  $W_{iN}$  was substituted with  $\tilde{W}_{iN}$

$$Y_{iN} = (a_i, b_i)\tilde{W}_{iN} + \Theta_{iN}. \quad (8)$$

Consequently, in (8) the disturbance

$$\Theta_{iN} = [\theta_i^{(1)}, \theta_i^{(2)}, \dots, \theta_i^{(N)}]$$

appears instead of  $\xi_{iN}$ . The situation is similar to the problem of identification of the simple linear dynamics with autoregression (see example in the Appendix). It was shown in [3], that because of correlation between the elements of  $\Theta_{iN}$  and  $\tilde{W}_{iN}$ , the estimation error

$$(\hat{a}_i^{l.s.}, \hat{b}_i^{l.s.}) - (a_i, b_i) = \Theta_{iN} \tilde{W}_{iN} \left( \tilde{W}_{iN} \tilde{W}_{iN}^T \right)^{-1}$$

does not tend to zero, as  $N \rightarrow \infty$ .

#### IV. INSTRUMENTAL VARIABLES APPROACH

To solve the problem shown in Remark 1 we propose the analogous strategy as for the SISO dynamic system identification (see e.g. [2] and [7]), i.e., generalization of (7) to the following form

$$(\widehat{a}_i^{i.v.}, \widehat{b}_i^{i.v.}) = Y_{iN} \Psi_{iN}^T \left( \widetilde{W}_{iN}^T \Psi_{iN}^T \right)^{-1}, \quad (9)$$

where

$$\Psi_{iN} = [\psi_i^{(1)}, \psi_i^{(2)}, \dots, \psi_i^{(N)}]$$

is the additional matrix of instrumental variables, of the same dimensions as  $\widetilde{W}_{iN}$ , i.e.

$$\psi_i^{(k)} = \left( \psi_{i,1}^{(k)}, \psi_{i,2}^{(k)} \right)^T.$$

We impose the following two conditions on  $\Psi_{iN}$ :

(C1) The instrumental variables  $\psi_{i,1}$ , and  $\psi_{i,2}$  are correlated with the input  $u_i$ , such that

$$\frac{1}{N} \widetilde{W}_{iN}^T \Psi_{iN}^T = \frac{1}{N} \sum_{k=1}^N \psi_i^{(k)} \widetilde{w}_i^{(k)T} \rightarrow E \psi_i \widetilde{w}_i^T$$

with probability 1 as  $N \rightarrow \infty$ , and the limit matrix  $E \psi_i \widetilde{w}_i^T$  is of full rank.

(C2) Simultaneously,  $\psi_{i,1}$ , and  $\psi_{i,2}$  are *not* correlated with the aggregated output noise  $\theta_i$ , i.e.,

$$\Psi_{iN} = L_i E N,$$

where

$$L_i = \begin{bmatrix} \Gamma_i & 0 \\ I_i & 0 \end{bmatrix},$$

and

$$I_i = [0, \dots, 0, 1, 0, \dots, 0].$$

*Theorem 2:* If the instrumental variables matrix  $\Psi_{iN}$  fulfils (C1) and (C2) then

$$(\widehat{a}_i^{i.v.}, \widehat{b}_i^{i.v.}) \rightarrow (a_i, b_i) \quad (10)$$

with probability 1, as  $N \rightarrow \infty$ .

*Proof:* The estimation error has the form

$$\begin{aligned} \Delta &= (\widehat{a}_i^{i.v.}, \widehat{b}_i^{i.v.}) - (a_i, b_i) = \\ &= \Theta_{iN} \Psi_{iN}^T \left( \widetilde{W}_{iN} \Psi_{iN}^T \right)^{-1} = \\ &= \left( \frac{1}{N} \Theta_{iN} \Psi_{iN}^T \right) \left( \frac{1}{N} \widetilde{W}_{iN} \Psi_{iN}^T \right)^{-1} \end{aligned} \quad (11)$$

where

$$\widetilde{W}_{iN} = F_i E N$$

and

$$F_i = \begin{bmatrix} H_i K & H_i G \\ I_i & 0 \end{bmatrix}.$$

Since, according to (C1) and (C2) it holds that

$$\begin{aligned} \frac{1}{N} \widetilde{W}_{iN} \Psi_{iN}^T &\rightarrow \begin{bmatrix} \text{cov}(\widetilde{x}_i, \psi_{i,1}) & \text{cov}(u_i, \psi_{i,1}) \\ \text{cov}(\widetilde{x}_i, \psi_{i,2}) & \text{cov}(u_i, \psi_{i,2}) \end{bmatrix}, \\ \frac{1}{N} \Theta_{iN} \Psi_{iN}^T &\rightarrow 0, \end{aligned}$$

from the Slutsky theorem we conclude (10). ■

In real applications the procedure of  $\Psi_{iN}$ -generation is of fundamental meaning. Let us introduce the quality index of instrumental variables

$$Q(\Psi_{iN}) = \|\Delta(\Psi_{iN})\| = \lambda_{\max}(\Delta(\Psi_{iN})\Delta^T(\Psi_{iN})). \quad (12)$$

The following theorem holds.

*Theorem 3:* The optimal instruments with respect to the value of  $Q(\Psi_{iN})$  has the form

$$\psi_i^* = \bar{w}_i = (\bar{x}_i, u_i)^T, \text{ where } \bar{x}_i = E(x_i|u) = H_i K u, \quad (13)$$

i.e.,

$$\Gamma_i = H_i K.$$

*Proof:* It is obvious that

$$y_i = (a_i, b_i) \widetilde{w}_i + \theta_i = (a_i, b_i) \bar{w}_i + a_i H_i G \theta + \theta_i = (a_i, b_i) \bar{w}_i + z_i,$$

where  $z_i = a_i H_i G \theta + \theta_i$  is a zero-mean disturbance, *uncorrelated* with the elements of the 'expected' input vector  $\bar{w}_i$ . ■

According to (11) we obtain that

$$\begin{aligned} \Delta(\Psi_{iN})\Delta^T(\Psi_{iN}) &= \frac{1}{\sqrt{N}} \Theta_{iN} \frac{1}{\sqrt{N}} \Psi_{iN}^T \\ &\cdot \left( \frac{1}{N} \widetilde{W}_{iN} \Psi_{iN}^T \right)^{-1} \left( \frac{1}{N} \Psi_{iN} \widetilde{W}_{iN}^T \right)^{-1} \\ &\cdot \frac{1}{\sqrt{N}} \Psi_{iN} \frac{1}{\sqrt{N}} \Theta_{iN}^T, \end{aligned}$$

and making use of the property that

$$\lambda_{\max}(\Delta(\Psi_{iN})\Delta^T(\Psi_{iN})) = \lambda_{\max}(\Delta^T(\Psi_{iN})\Delta(\Psi_{iN})),$$

for  $N$  large and  $\Psi_{iN}^* = \bar{W}_{iN} = [\bar{w}_i^{(1)}, \bar{w}_i^{(2)}, \dots, \bar{w}_i^{(N)}]$  we simply get

$$Q(\Psi_{iN}^*) = \lambda_{\max} \left( \frac{1}{N} \bar{W}_{iN} \bar{W}_{iN}^T \right)^{-1} \text{var} \theta_i.$$

Under Lemma 6 in [4], for each  $\Psi_{iN}$  it holds that

$$Q(\Psi_{iN}^*) \leq Q(\Psi_{iN})$$

with probability 1.

Since the matrix  $K$  is unknown, the result (13) is not constructive, but gives the general concept of using the estimates of the noise-free interactions  $x_i$ . In the simulation we used the approximation of  $K$  obtained by the least square method and implemented the recursive version of the algorithm

$$\begin{aligned} (\widehat{a}_i^{i.v.}, \widehat{b}_i^{i.v.})_{(k)} &= (\widehat{a}_i^{i.v.}, \widehat{b}_i^{i.v.})_{(k-1)} + \\ &+ \left[ y_i^{(k)} - (\widehat{a}_i^{i.v.}, \widehat{b}_i^{i.v.})_{(k-1)} \widetilde{w}_i^{(k)} \right] \psi_i^{(k)T} P_{i,k}, \end{aligned}$$

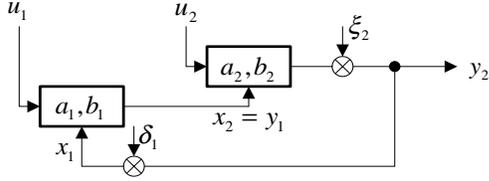


Fig. 2. The two-element cascade system with feedback

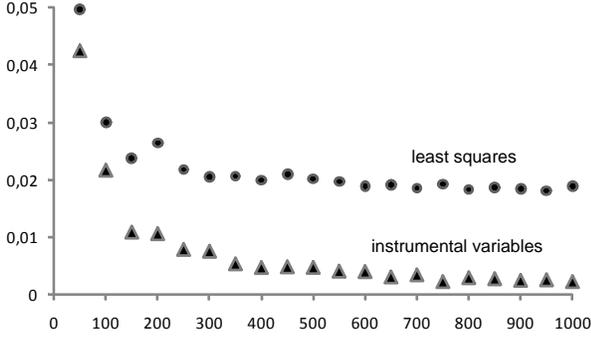


Fig. 3. The estimation error  $\Delta$  versus number of measurements  $N$

where

$$P_{i,k} = \frac{P_{i,k-1} - P_{i,k-1} \tilde{w}_i^{(k)} \psi_i^{(k)T} P_{i,k-1}}{1 + \psi_i^{(k)T} P_{i,k-1} \tilde{w}_i^{(k)}}$$

## V. SIMULATION EXAMPLE

In this section we present the performance of the algorithm on the example of the simple, two-element cascade system with feedback (see Fig. 2). We set  $(a_1, b_1) = (1, 1)$ , and  $(a_2, b_2) = (2, 2)$  and the interconnections are coded as follows

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The system is excited by two independent uniformly random processes  $u_1, u_2 \sim U(0, 1)$ , and disturbed by zero-mean noises  $\delta_1, \xi_2 \sim U(-0.1, 0.1)$ . For both elements, the instrumental variables estimates are computed and compared with the least squares results.

In Fig. 3 we present the Euclidean norm of the error  $\Delta$  for both algorithms.

## VI. SUMMARY

The idea of instrumental variables estimate was successfully generalized for the complex, interconnected systems. Since the production and transportation systems usually work in steady state, we limited ourselves to the static blocks. However, generalization for the FIR linear dynamic components seems to be quite simple. For details, we refer the reader to [6], where similar approach was applied for identification of interconnected Hammerstein systems (see Fig. 4). In the contrary to the traditional least squares approach, the proposed algorithm recovers true parameters of subsystems. General

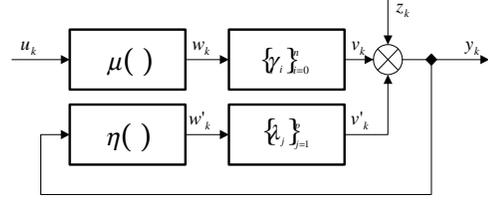


Fig. 4. The Narmax system

conditions are imposed on the instrumental variables for to estimate to be consistent, and then the form of optimal values of instruments is shown. We emphasize that the method works for any structure of the system and for any distribution of the random noises.

## VII. APPENDIX

In this section the subscript  $k$  is used for the time instant. Consider the simple AR(1) linear dynamic object with the input  $u_k$  and the output  $v_k$ , disturbed by the random process  $\varepsilon_k$ , i.e.

$$v_k = bu_k + av_{k-1}, \text{ and } y_k = v_k + \varepsilon_k.$$

Since the noise-free output  $v_k$  is unknown, we must base on the difference equation describing dependence between  $u_k$  and the measured output

$$y_k = bu_k + ay_{k-1} + z_k = (a, b)\phi_k + z_k,$$

where  $\phi_k = (y_{k-1}, u_k)^T$  and the resulting disturbance  $z_k = \varepsilon_k - a\varepsilon_{k-1}$  is obviously correlated with  $y_{k-1}$ , included in the generalized 'input'  $\phi_k$ .

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