Proper local minimum

Function \( f(x) \) has a proper local minimum in point \( \lambda \), if there exists \( \delta > 0 \)
such that for \( \forall x \in X \)
\[
f(\lambda) \leq f(x)
\]
where:
\[
E = X \cap A_l
\]
\[
A_l = \{ x : 0 < \| x - \lambda \| < \delta \}
\]

Weak local minimum

Function \( f(x) \) has a weak local minimum in point \( \lambda \), if there exists \( \delta > 0 \)
such that, for \( \forall x \in E \)
\[
f(\lambda) \leq f(x)
\]
where:
\[
E = X \cap A_l
\]
\[
A_l = \{ x : 0 < \| x - \lambda \| < \delta \}
\]

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Local minimum and global minimum of a function \( f(x) \)

The point \( \lambda \) determines a local minimum of a function \( f(x) \) in \( \mathbb{R}^n \) space.
When: \( \forall x \in X \), \( f(\lambda) \leq f(x) \)

The point \( \lambda \) determines a global minimum of a function \( f(x) \) in \( \mathbb{R}^n \) space,
when \( \forall x \in X \) \( f(\lambda) \leq f(x) \)

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The Weierstrass extreme value theorem

A continuous function \( f(x) \) defined on a compact set of admissible solutions (on a closed and bounded set) is bounded on this set and obtains its bounds.: two points exist, \( x_i, x_j \in X \)
such that for each \( x \in X \)
the following relation is valued:
\[
f(x_i) \leq f(x) \leq f(x_j)
\]

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Definition of a convex set

In Euclidean space the set \( X \subset \mathbb{R}^n \) is convex, if for every pair of points \( x^1, x^2 \in X \)
within the set, every point on the straight line segment that joins the pairs of points is also within this set \( X \):
\[
x = \{ x : x = \lambda x^1 + (1 - \lambda)x^2, 0 \leq \lambda \leq 1 \}
\]

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Definition of a convex function

In Euclidean space the set \( X \subset \mathbb{R}^n \) is convex. Function \( f(x) : X \to \mathbb{R} \) will be a convex function, if for every pair of points \( x^1, x^2 \in X \)
and each \( \lambda \in [0, 1] \) the following inequality is fulfilled:
\[
f(\lambda x^1 + (1 - \lambda)x^2) \leq \lambda f(x^1) + (1 - \lambda)f(x^2)
\]

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Definition of a strictly convex function

In Euclidean space the set \( X \subset \mathbb{R}^n \) is convex. Function \( f(x) : X \to \mathbb{R} \)
will be strictly convex, if for every pair of points \( x^1, x^2 \in X \)
and each \( \lambda \in [0, 1] \) the following inequality is fulfilled:
\[
f(\lambda x^1 + (1 - \lambda)x^2) < \lambda f(x^1) + (1 - \lambda)f(x^2)
\]

Definition of a concave function

In Euclidean space the set \( X \subset \mathbb{R}^n \) is convex. Function \( f(x) : X \to \mathbb{R} \)
will be a concave function, if for every pair of points \( x^1, x^2 \in X \)
and each \( \lambda \in [0, 1] \) the following inequality is fulfilled:
\[
f(\lambda x^1 + (1 - \lambda)x^2) \geq \lambda f(x^1) + (1 - \lambda)f(x^2)
\]

Collorary: Function \( f(x) \) is concave if and only if function \(-f(x)\) is convex.

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Is a function \( f(x) \) – a convex function

A matrix \( A \) (dim n×n) is called hessian, when its elements are partial derivative of second order of a function \( f(x) \):
\[
A(x) \begin{bmatrix} f''(x) \\ \nabla f(x) \end{bmatrix}
\]

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Definition of a positive half-definite matrix $A$

A matrix $A$ is a positive half-definite matrix, when for each $x \in \mathbb{R}^n$

$\langle x, Ax \rangle \geq 0$

Definition of a positive definite matrix $A$

A matrix $A$ is a positive definite matrix, when for each $x \in \mathbb{R}^n$

$\langle x, Ax \rangle > 0$

Sylwester Criterion –

Practical testing of a function convexity

A square symmetric matrix $A$ is positive half-definite matrix, if and only if:

$s_{ij} \geq 0, \begin{bmatrix} s_{11} & \ldots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \ldots & s_{nn} \end{bmatrix} \geq 0$

A square symmetric matrix $A$ is a positive definite matrix, if and only if:

$s_{ij} > 0, \begin{bmatrix} s_{11} & \ldots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \ldots & s_{nn} \end{bmatrix} > 0$

Fig.1  The convex set $X$

Convex function $f(x)$

Fig.2 The non-convex set of $X$

Function $f(x)$ – a convex function.

Fig.3 The convex set of $X$

Function $f(x)$ – non-convex function

Fig.4 The non-convex set of $X$

Function $f(x)$ – non-convex function.
Convex function

**Theorem 2**
Let the set $X \subseteq \mathbb{R}^n$ be convex and let function $f(x)$ be differentiable. Then function $f(x)$ will be convex, if and only if

$$f(x) > f(x_0) + \nabla f(x_0)^T (x-x_0)$$


**Theorem 3**
Let a function $f_1: X \rightarrow \mathbb{R}$ be convex. Then for each real value $x$, the set

$$X_{\lambda} = \{ x : f(x) \leq \lambda \}$$

is convex.

**Theorem 4**
Let $f_i: X \rightarrow \mathbb{R}$ for $i=1,...,k$ be convex functions and if scalar value $u_i \geq 0$ for $i=1,...,k$ then function

$$f(x) = \sum_{i=1}^{k} u_i f_i(x)$$

is a convex function.

**Theorem 5**
The local minimum of a convex function on a convex set $X \subseteq \mathbb{R}^n$ of admissible solutions, is the global minimum of that function on the set $X$.

Proof:
Let the function $f(x)$ has its local minimum in point $x^*$, it means that there exists such $\epsilon > 0$, that:

$$f(x^*) = \min f(x)$$

where $V = \{ x \in X : x^T x \leq \epsilon \}$.

Assume, that $x^* \in X$, $x^* \neq \lambda x$. Let us consider $0 < \lambda < 1$ and

$$\lambda x + (1-\lambda) x^* \in V$$

In view of convexity of a function $f(x)$:

$$\lambda f(x) + (1-\lambda) f(x^*) \geq f(\lambda x + (1-\lambda) x^*)$$

which completes the proof.

**Theorem 6**
Strictly convex function $f(x)$ defined on a convex set of admissible solutions has at most one minimum on this set.

**Example**
Let $c \in \mathbb{R}^n$, $d \in \mathbb{R}$. Try to show, that function $f(x)$ defined as follows:

$$f(x) = c^T x + d$$

is simultaneously convex and concave function on a set of $\mathbb{R}^n$.

**Theorem 7**
When

$$f(x): \mathbb{R}^n \rightarrow \mathbb{R}$$

is concave function and $X$ is a bounded, closed and convex set, there exists extreme point of a set $X$, where the function $f(x)$ achieves the minimum value on this set.

**Colorary**
The linear function $f(x)$ defined on a polyhedron (the convex set of admissible solutions) achieves its bounds on extreme points of a polyhedron.